Learning Invariant Riemannian Geometric Representations Using Deep Nets

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Manifold-Aware Regression

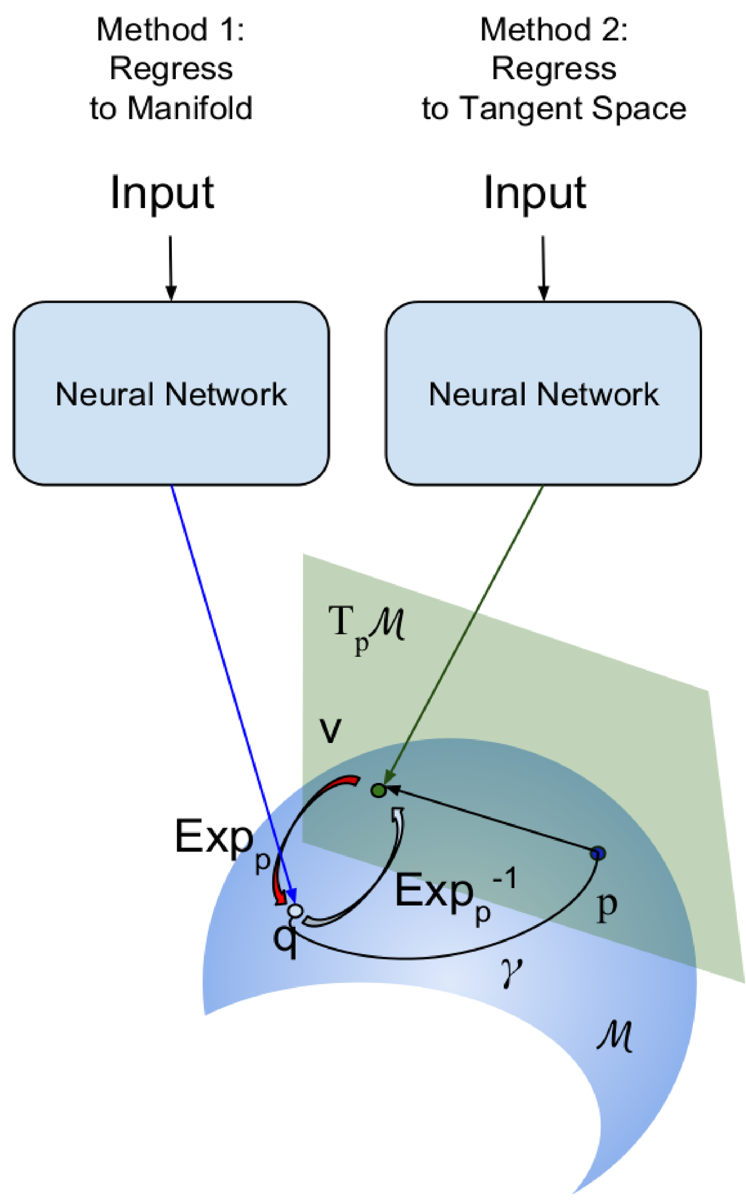
- Manifold-valued data representations in computer vision - invariant representations (illumination-invariant subspaces), task-determined manifold representations (saliency detection, classification)
- We are interested in the problem of learning a non-linear function to regress a manifold-valued response, given a vector-valued input

$$f : \mathbb{R}^N \to \mathcal{M}$$

- Prior work uses hand-crafted features e.g. Fletcher [1], Banerjee et al. [2]
- We use a neural network architecture as the features are learned directly from data
- However, using a neural network introduces new challenges : loss function and manifold constraints

Two Approaches for Neural Networks

- For "simple" manifolds, we can employ the neural net to directly map to the manifold using the geodesic distance as the loss function. Manifold constraints can be satisfied exactly using differentiable layers e.g. (n-1)-sphere
- Learn the network to map to the tangent space of the manifold. The Euclidean loss on the tangent space can be used as the loss function. For Grassmann and Stiefel manifolds, tangent space constraints are easier to satisfy. Exponential map is used to get the desired point on manifold



Grassmann Geometry

- The Grassmann manifold is the set of all d-dimensional subspaces of $\mathbb{R}^N$

$$\mathcal{G}_{n,d} = \{[\mathbf{U}]\}, [\mathbf{U}] = \{\mathbf{UQ} | \mathbf{U} \in \mathbb{R}^{n\times d}, \mathbf{U}^T\mathbf{U} = \mathbf{I}, \mathbf{Q} \text{ is orthogonal}\}$$

- The tangent vectors at the identity matrix have nice structure

$$\mathbf{X} = \begin{bmatrix} \mathbf{0}_d & \mathbf{A} \\ -\mathbf{A}^T & \mathbf{0}_{n-d} \end{bmatrix}, \mathbf{A} \in \mathbb{R}^{d\times(n-d)}$$

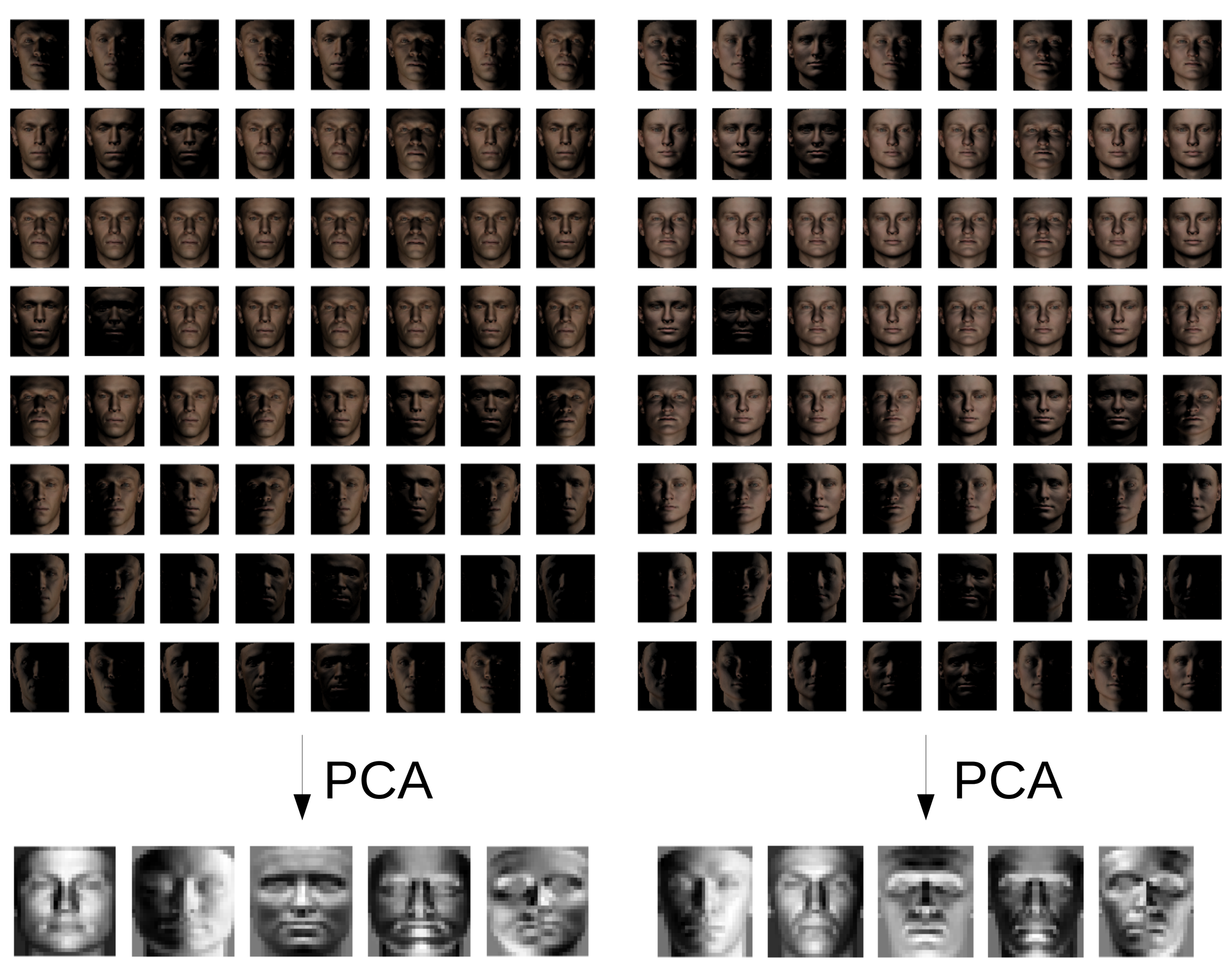
- The geodesic distance between two subspaces is given by

$$D_G(\mathbf{U}_1, \mathbf{U}_2) = \sqrt{\sum_{i=1}^{d} \theta_i^2}$$

$$\mathbf{W}\cos(\Theta)\mathbf{V}^T = \mathrm{svd}(\mathbf{U}_1^T\mathbf{U}_2)$$

Experiments on Grassmann Manifold: Single Face to Illumination Subspace

- We design an ill-posed problem for illustration of regression on the Grassmannian: Given a face image of an unseen subject under unknown illumination, predict the illumination subspace



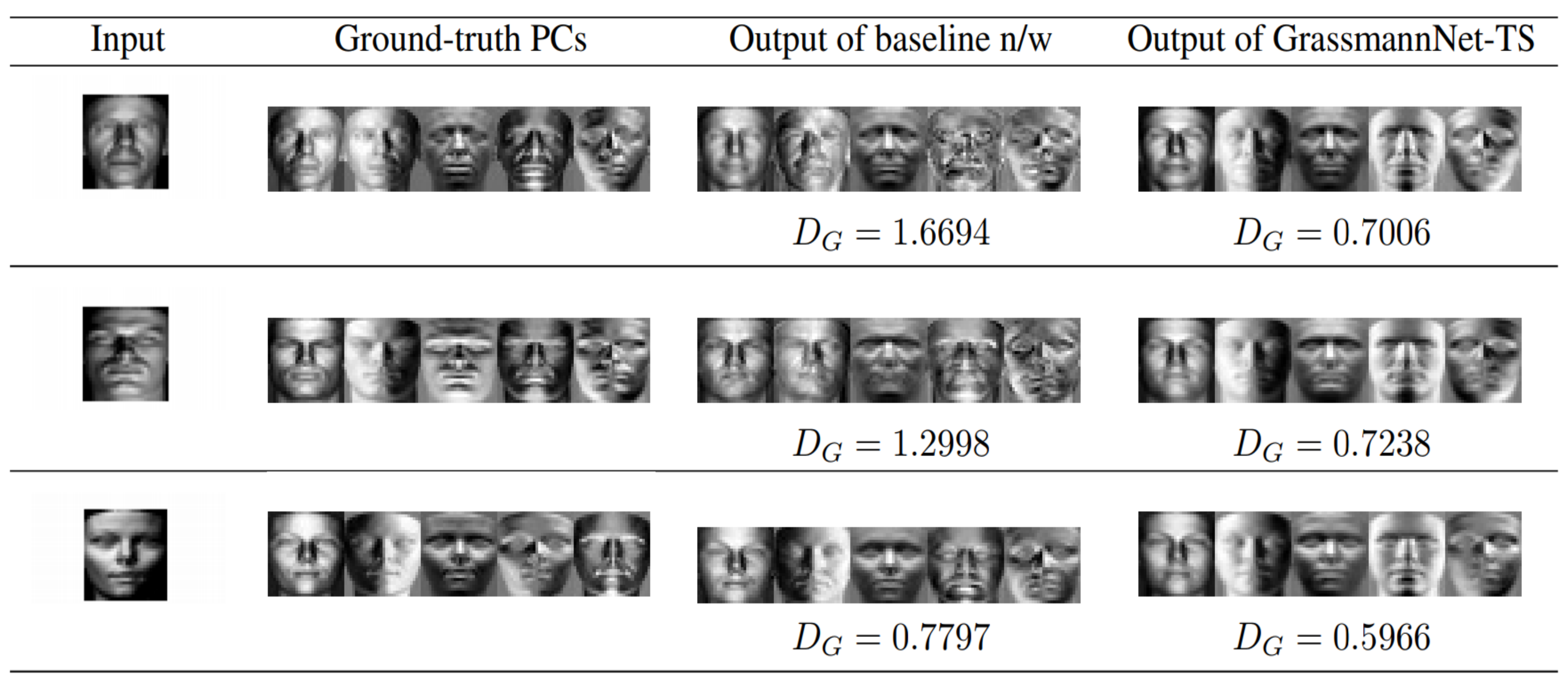
- Baseline: Direct regression on Grassmannian using Euclidean loss function (ignoring geometry)

$$L_{euc} = ||\mathbf{U} - \hat{\mathbf{U}}||_F$$

- Mapping via the tangent space with Euclidean loss on the tangent space and using exponential map on the network output to find the corresponding subspace

$$L_{gr} = ||\mathbf{A} - \hat{\mathbf{A}}||_F$$

- We experiment with two poles – Frechet mean and PCA of entire training set



Subspace Dimension	Baseline	GrassmannNet-TS: PCA of training set	GrassmannNet-TS: Frechet Mean
3	0.6613	0.3991	0.3953
4	1.0997	0.5489	0.5913
5	1.4558	0.8694	0.6174

Ongoing/Future Work

- Experiments on image relighting and illumination-invariant single-image face recognition show promising results
- Experiments on the unit hypersphere: Multi-classification can be posed as regression on the unit hypersphere using the square-root parameterization
- Initial experiments on MNIST and CIFAR-10 indicate the geodesic loss on the sphere performs better than the cross-entropy loss
- Extending to other interesting manifolds such as SPDs for applications in diffusion tensor imaging
- Extend current ideas to multiple tangent spaces for data with more variance on the manifold

[1] Fletcher, Thomas, "Geodesic regression on Riemannian manifolds." *Proc. 3rd Intl. Workshop on Math. Foundations of Comp. Anatomy – Geometrical and Statistical Methods for Modelling Biological Shape Variability.* 2011
[2] Banerjee Momami et al., "Non-linear regression on Riemannian manifolds and its application to neuro-image analysis." *Intl. Conf. On Medical Image Computing and Computer-Assisted Intervention*, Springer, Cham, 2015